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# $J$-FILTERS OF BOUNDED LATTICES

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Abstract. Let $\mathcal{L}$ be a bounded distributive lattice and $J(\mathcal{L})$ denote the Jacobson radical of $\mathcal{L}$. Similar to the definition of $J$-ideals of commutative rings, we introduce and study $J$-filters of lattices. A proper filter $F$ of $\mathcal{L}$ is called a $J$-filter if whenever $x \vee y \in F$ with $x \notin J(\mathcal{L})$, then $y \in F$ for every $x, y \in \mathcal{L}$. The main purpose of this paper is devoted to extend the notion of $J$-ideal property in commutative rings to $J$-filter property in lattices.

## 1. Introduction

All lattices considered in this paper are assumed to have a least element denoted by 0 and a greatest element denoted by 1, in other words they are bounded. As algebraic structures, lattices are undoubtedly a natural choice of generalizations of rings. In structure, lattices lie between semigroups and rings. The main aim of this article is that of extending some results obtained for ring theory to lattice theory. The main difficulty is figuring out what additional hypothesis the lattice or filter must satisfy to get similar results. Nevertheless, growing interest in developing the algebraic theory of lattices can be found in several papers and books (see for instance [3, 4, 6, 7, 8, 9]).

Over the years, several types of ideals have been developed in order to let us fully understand the structures of rings in general. The notion of prime ideal plays a key role in the theory of commutative algebra, and it has been widely studied. See, for example, [1, 5]. Recall from [2], a prime ideal $P$ of $R$ is a proper ideal having the property that $ab \in P$ implies either $a \in P$ or $b \in P$ for each $a, b \in R$. In [12], Mohamadian defined a proper ideal $I$ of $R$ as an $r$-ideal if whenever $a, b \in R$ with $ab \in I$ and $\operatorname{ann}(a) = 0$ imply that $b \in I$, where $\operatorname{ann}(a) = \{r \in R : ra = 0\}$. He investigated the behavior of $r$-ideals and compare them with other classical ideals such as prime and maximal ideals. In [14], Tekir et al., defined and studied some subclass of $r$-ideals, namely, the class of $n$-ideals. A proper ideal $I$ of a ring $R$ is called an $n$-ideal if whenever $a, b \in R$ with $ab \in I$ and $a \notin \sqrt{0}$, then $b \in I$. Khashan and Bani-Ata generalized the concept of $n$-ideals in [11]. A proper ideal $I$ of a ring $R$ is called a $J$-ideal if whenever $a, b \in R$ with $ab \in I$ and $a \notin J(R)$, then $b \in I$. Let $\mathcal{L}$ be a bounded distributive lattice. Our objective in this paper is to extend the notion of $J$-ideal property in commutative rings to $J$-filter property in the lattices. Among many other results in this paper, the first, preliminaries section contains elementary observations needed later on.

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Section 3 is dedicated to the investigate the basic properties of $J$-filters. A proper filter $F$ of $\mathcal{L}$ is called a $J$-filter if whenever $x \vee y \in F$ with $x \notin J(\mathcal{L})$, then $y \in F$ for every $x, y \in \mathcal{L}$. At first, it is shown (Proposition 3.2) that if $F$ is a $J$-filter of a lattice $\mathcal{L}$, then $F \subseteq J(\mathcal{L})$ and provide an example (Example 3.3) shows that we can find a filter $F$ of a lattice $\mathcal{L}$ with $F \subseteq J(\mathcal{L})$ which is not a $J$-filter . In Theorem 3.4, we characterize lattices in which every proper filter is a $J$-filter. We give some examples to show that in general $J$-filters are not comparable with prime, maximal and $r$-filters (Example 3.5 and example 3.9). However, if a lattice $\mathcal{L}$ is presimplifiable (Definition 3.6), then we prove that any $r$-filter in $\mathcal{L}$ is a $J$-filter (Proposition 3.7). In Theorem 3.11, we give three other characterizations of $J$-filters. We investigate the relations between $J$-filters and prime filters. We observe that every maximal $J$-filter in a lattice is a prime filter (Proposition 3.13). We clarify that a filter of a lattice contained in the Jacobson radical is a $J$-filter if and only if it is $J$-primary (Proposition 3.19). It is shown that if $P \in \min(F)$, where $F$ is a $J$-filter of $\mathcal{L}$, then $P$ is a $J$-filter (Theorem 3.22). Finally, we naturally extend the Prime Avoidance Lemma to $J$-filters (Theorem 3.24).

In Section 4, we investigate the properties of $J$-filters similar to prime filters. In particular, we investigate the behavior of $J$-filters under homomorphism, in factor lattices, $\vee$-lattices, minimal $J$-filters over a filter $F$ and in cartesian products of lattices (see Theorem 4.2, Proposition 4.4, Theorem 4.5, Theorem 4.9, Theorem 4.11, Theorem 4.12, Theorem 4.15 and Theorem 4.17).

## 2. Preliminaries

A *poset* $(\mathcal{L}, \leq)$ is a *lattice* if $\sup\{a, b\} = a \vee b$ and $\inf\{a, b\} = a \wedge b$ exist for all $a, b \in \mathcal{L}$ (and call $\wedge$ the *meet* and $\vee$ the *join*). A lattice $\mathcal{L}$ is *complete* when each of its subsets $X$ has a least upper bound and a greatest lower bound in $\mathcal{L}$. Setting $X = \mathcal{L}$, we see that any nonvoid complete lattice contains a least element 0 and greatest element 1 (in this case, we say that $\mathcal{L}$ is a lattice with 0 and 1). A lattice $\mathcal{L}$ is called a *distributive lattice* if $(a \vee b) \wedge c = (a \wedge c) \vee (b \wedge c)$ for all $a, b, c$ in $\mathcal{L}$ (equivalently, $\mathcal{L}$ is distributive if $(a \wedge b) \vee c = (a \vee c) \wedge (b \vee c)$ for all $a, b, c$ in $\mathcal{L}$). A non-empty subset $F$ of a lattice $\mathcal{L}$ is called a *filter*, if for $a \in F$, $b \in \mathcal{L}$, $a \leq b$ implies $b \in F$, and $x \wedge y \in F$ for all $x, y \in F$ (so if $\mathcal{L}$ is a lattice with 0 and 1, then $1 \in F$, $\{1\}$ is a filter of $\mathcal{L}$ and $0 \in F$ if and only if $F = \mathcal{L}$). A proper filter $F$ of $\mathcal{L}$ is called *prime* if $x \vee y \in F$, then $x \in F$ or $y \in F$. A proper filter $F$ of $\mathcal{L}$ is said to be *maximal* if $G$ is a filter in $\mathcal{L}$ with $F \subsetneqq G$, then $G = \mathcal{L}$. We define the *Jacobson radical* of $\mathcal{L}$, denoted by $J(\mathcal{L})$, to be the intersection of all the maximal filters of $\mathcal{L}$. The set of all maximal filters of $\mathcal{L}$ is denoted $\mathrm{Max}(\mathcal{L})$.

Let $D$ be subset of a lattice $\mathcal{L}$. Then the filter generated by $D$, denoted by $T(D)$, is the intersection of all filters that is containing $D$. A filter $F$ is called finitely generated if there is a finite subset $D$ of $F$ such that $F = T(D)$. A lattice $\mathcal{L}$ with 1 is called $\mathcal{L}$-domain if $a \vee b = 1$ $(a, b \in \mathcal{L})$, then $a = 1$ or $b = 1$ (so $\mathcal{L}$ is $\mathcal{L}$-domain if and only if $\{1\}$ is a prime filter of $\mathcal{L}$). A lattice $\mathcal{L}$ is called *local* if it has exactly one maximal filter that contains all proper filters. If $x \in \mathcal{L}$, then a *complement* of $x$ in $\mathcal{L}$ is an element $y \in \mathcal{L}$ such that $x \vee y = 1$ and $x \wedge y = 0$. The lattice $\mathcal{L}$ is *complemented* if every element of $\mathcal{L}$ has a complement in $\mathcal{L}$. If $\mathcal{L}$ and $\mathcal{L}'$ are lattices, then a *lattice homomorphism* $f : \mathcal{L} \to \mathcal{L}'$ is a map from $\mathcal{L}$ to $\mathcal{L}'$ satisfying $f(x \vee y) = f(x) \vee f(y)$ and $f(x \wedge y) = f(x) \wedge f(y)$ for $x, y \in \mathcal{L}$. A filter $G$ of $\mathcal{L}$ is called *small* in $\mathcal{L}$, written $G \ll \mathcal{L}$, if for every filter $H$ of $\mathcal{L}$, the equality $G \wedge H = \mathcal{L}$ implies $H = \mathcal{L}$. For undefined notations or

terminologies in lattice theory, we refer the reader to [3, 4]. First we need the following easy observation proved in [6, 7, 8, 9].

**Lemma 2.1.** *Let $\mathcal{L}$ be a lattice.*

*(1) A non-empty subset $F$ of $\mathcal{L}$ is a filter of $\mathcal{L}$ if and only if $x \vee z \in F$ and $x \wedge y \in F$ for all $x, y \in F$, $z \in \mathcal{L}$. Moreover, since $x = x \vee (x \wedge y)$, $y = y \vee (x \wedge y)$ and $F$ is a filter, $x \wedge y \in F$ gives $x, y \in F$ for all $x, y \in \mathcal{L}$.*

*(2) If $F_1, \cdots, F_n$ are filters of $\mathcal{L}$ and $a \in \mathcal{L}$, then $\bigvee_{i=1}^{n} F_i = \{\bigvee_{i=1}^{n} a_i : a_i \in F_i\}$ and $a \vee F_i = \{a \vee a_i : a_i \in F_i\}$ are filters of $\mathcal{L}$ and $\bigvee_{i=1}^{n} F_i = \bigcap_{i=1}^{n} F_i$.*

*(3) Let $A$ be an arbitrary non-empty subset of $\mathcal{L}$. Then*

$$T(A) = \{x \in \mathcal{L} : \ a_1 \wedge a_2 \wedge \cdots \wedge a_n \leq x \text{ for some } a_i \in A \ (1 \leq i \leq n)\}.$$

*Moreover, if $F$ is a filter and $A$ is a subset of $\mathcal{L}$ with $A \subseteq F$, then $T(A) \subseteq F$, $T(F) = F$ and $T(T(A)) = T(A)$*

*(4) If $\mathcal{L}$ is distributive, $F, G$ are filters of $\mathcal{L}$ and $y \in \mathcal{L}$, then $(G :_{\mathcal{L}} F) = \{x \in \mathcal{L} : x \vee F \subseteq G\}$, $(F :_{\mathcal{L}} T(\{y\})) = (F :_{\mathcal{L}} y) = \{a \in \mathcal{L} : a \vee y \in F\}$ and $(1 :_{\mathcal{L}} y) = \{x \in \mathcal{L} : x \vee y = 1\}$ are filters of $\mathcal{L}$.*

*(5) If $\{F_i\}_{i \in \Delta}$ is a chain of filters of $\mathcal{L}$, then $\bigcup_{i \in \Delta} F_i$ is a filter of $\mathcal{L}$.*

*(6) If $\mathcal{L}$ is distributive, $G, F_1, \cdots, F_n$ are filters of $\mathcal{L}$, then $G \vee (\bigwedge_{i=1}^{n} F_i) = \bigwedge_{i=1}^{n} (G \vee F_i)$.*

*(7) If $\mathcal{L}$ is distributive and $F_1, \cdots, F_n$ are filters of $\mathcal{L}$, then $\bigwedge_{i=1}^{n} F_i = \{\wedge_{i=1}^{n} a_i : a_i \in F_i\}$ is a filter of $\mathcal{L}$ and $F_i \subseteq \bigwedge_{i=1}^{n} F_i$ for each $i$.*

*(8) If $f : \mathcal{L} \to \mathcal{L}'$ is a lattice homomorphism with $f(1) = 1$, then $\mathrm{Ker}(f) = \{x \in \mathcal{L} : f(x) = 1\}$ is a filter of $\mathcal{L}$.*

## 3. Characterization of $J$-Filters

In this section, we collect some basic properties concerning $J$-filters. We remind the reader with the following definition.

**Definition 3.1.** A proper filter $F$ of a lattice $\mathcal{L}$ is called a $J$-filter if whenever $x, y \in \mathcal{L}$ with $x \vee y \in F$ and $x \notin J(\mathcal{L})$, then $y \in F$.

**Proposition 3.2.** *If $F$ is a $J$-filter of a lattice $\mathcal{L}$, then $F \subseteq J(\mathcal{L})$.*

*Proof.* On the contrary, assume that $F \nsubseteq J(\mathcal{L})$. Then there exists $f \in F$ such that $f \notin J(\mathcal{L})$. Now $f = f \vee 0 \in F$ and $f \notin J(\mathcal{L})$ implies that $0 \in F$, a contradiction. Hence, $F \subseteq J(\mathcal{L})$, as needed. □

The following example shows that we can find a filter $F$ of a lattice $\mathcal{L}$ with $F \subseteq J(\mathcal{L})$ which is not a $J$-filter.

**Example 3.3.** Assume that $\mathcal{L} = \{0, a, b, c, 1\}$ is a lattice with the relations $0 \leq a \leq c \leq 1$, $0 \leq b \leq c \leq 1$, $a \vee b = c$ and $a \wedge b = 0$. An inspection will show that the nontrivial filters of $\mathcal{L}$ are $F_1 = \{1, a, c\}$, $F_2 = \{1, b, c\}$ and $F_3 = \{1, c\}$ with $F_3 \subseteq J(\mathcal{L}) = F_1 \cap F_2$. But $F_3$ is not a $J$-filter since $a \vee b = c \in F_3$ with $a \notin J(\mathcal{L})$ and $b \notin F_3$.

In the following theorem, we characterize lattices in which every proper filter is a $J$-filter.

**Theorem 3.4.** *If $\mathcal{L}$ is a lattice, then the following statements are equivalent:*

*(1) $\mathcal{L}$ is a local lattice;*

*(2) Every proper filter of $\mathcal{L}$ is a $J$-filter;*

*(3) Every proper principal filter of $\mathcal{L}$ is a $J$-filter.*

*Proof.* (1) $\Rightarrow$ (2) Assume that $F$ is a proper filter of a local lattice $\mathcal{L}$ with unique maximal filter $\mathbf{m}$ and let $x, y \in \mathcal{L}$ with $x \vee y \in F$ and $x \notin J(\mathcal{L}) = \mathbf{m}$. Then $\mathbf{m} \subsetneqq \mathbf{m} \wedge T(\{x\}) \subseteq \mathcal{L}$ gives $\mathbf{m} \wedge T(\{x\}) = \mathcal{L}$ which implies that $T(\{x\}) = \mathcal{L}$, as $\mathbf{m} \ll \mathcal{L}$ by [7, Lemma 3.1]. So $0 = x \vee a$ for some $a \in \mathcal{L}$; hence $y = 0 \vee y = a \vee x \vee y \in F$. Thus $F$ is a $J$-filter. The implication (2) $\Rightarrow$ (3) is clear

(3) $\Rightarrow$ (1) Suppose that every proper principal filter of $\mathcal{L}$ is a $J$-filter. Assume that $\mathbf{m}$ is a maximal filter of $\mathcal{L}$ and let $z \in \mathbf{m}$. Now, $T(\{z\})$ is a $J$-filter and $z \vee 0 \in T(\{z\})$. If $z \notin J(\mathcal{L})$, then $0 \in T(\{z\})$, a contradiction. Thus, $z \in J(\mathcal{L})$ and $\mathbf{m} = J(\mathcal{L})$ is the unique maximal filter of $\mathcal{L}$. Therefore, $\mathcal{L}$ is a local lattice. □

In the following example, it is shown that, in general, the class of $J$-filters is not comparable with the classes of prime filters and maximal filters.

**Example 3.5.** (1) Let $D = \{a, b, c\}$. Then $\mathcal{L} = \{X : X \subseteq D\}$ forms a distributive lattice under set inclusion with greatest element $D$ and least element $\emptyset$ (note that if $x, y \in \mathcal{L}$, then $x \vee y = x \cup y$ and $x \wedge y = x \cap y$). It can be easily seen that the set of all proper filters $\mathcal{L}$ is $\{\{D\}, F_1, F_2, F_3, F_4, F_5, F_6\}$, where $F_1 = \{D, \{a, b\}\}$, $F_2 = \{D, \{a, c\}\}$, $F_3 = \{D, \{b, c\}\}$,

$$F_4 = \{D, \{a, c\}, \{a, b\}, \{a\}\},$$

$F_5 = \{D, \{b, c\}, \{a, b\}, \{b\}\}$ and $F_6 = \{D, \{a, c\}, \{c, b\}, \{c\}\}$ with $J(\mathcal{L}) = F_4 \cap F_5 \cap F_6 = \{1\}$. Then the filter $F_4$ is a maximal (prime) filter. On the other hand, $F_4$ is not a $J$-filter of $\mathcal{L}$, as $F_4 \nsubseteq J(\mathcal{L})$.

(2) The collection of ideals of $\mathbb{Z}$, the ring of integers, form a lattice under set inclusion which we shall denote by $\mathcal{L}$ with respect to the following definitions: $m\mathbb{Z} \vee n\mathbb{Z} = (m, n)\mathbb{Z}$ and $m\mathbb{Z} \wedge n\mathbb{Z} = [m, n]\mathbb{Z}$ for all ideals $m\mathbb{Z}$ and $n\mathbb{Z}$ of $\mathbb{Z}$, where $(m, n)$ and $[m, n]$ are greatest common divisor and least common multiple of $m, n$, respectively. Note that $\mathcal{L}$ is a distributive complete lattice with least element the zero ideal and the greatest element $\mathbb{Z}$. By [6, Theorem 2.9 (ii)], $\mathcal{L} \setminus \{0\}$ is the only maximal filter of $\mathcal{L}$ and so $\mathcal{L}$ is a local lattice. It follows from Theorem 3.4 that every proper filter of $\mathcal{L}$ is a $J$-filter. Consider the filter $F = \{\mathbb{Z}, 2\mathbb{Z}, 4\mathbb{Z}, \cdots\}$. Since $5\mathbb{Z} \vee 7\mathbb{Z} = \mathbb{Z} \in F$ with $5\mathbb{Z}, 7\mathbb{Z} \notin F$, we infer that $F$ is not a prime (maximal) filter.

An element $x$ of $\mathcal{L}$ is called *identity join* of a lattice $\mathcal{L}$, if there exists $1 \neq y \in \mathcal{L}$ such that $x \vee y = 1$. The set of all identity joins of a lattice $\mathcal{L}$ is denoted by $\mathrm{I}(\mathcal{L})$.

**Definition 3.6.** (1) A proper filter $F$ of a lattice $\mathcal{L}$ is called an $r$-filter if whenever $a, b \in \mathcal{L}$ with $a \vee b \in F$ and $(1 :_{\mathcal{L}} a) = \{1\}$, then $b \in F$.

(2) A proper filter $F$ of $\mathcal{L}$ is called an 1-filter if whenever $a, b \in \mathcal{L}$ with $a \vee b \in F$ and $a \neq 1$, then $b \in F$.

(3) A lattice $\mathcal{L}$ is called presimplifiable if $\mathrm{I}(\mathcal{L}) \subseteq J(\mathcal{L})$.

**Proposition 3.7.** *If $\mathcal{L}$ is a presimplifiable lattice, then the following hold:*

*(1) Every $r$-filter in $\mathcal{L}$ is a $J$-filter.*

*(2) if $a, b \in \mathcal{L}$ with $a \wedge b = 0$, then $F = (1 :_{\mathcal{L}} a) \wedge (1 :_{\mathcal{L}} b)$ is a $J$-filter.*

*Proof.* (1) Suppose $\mathcal{L}$ is presimplifiable and let $F$ be an $r$-filter of $\mathcal{L}$. Let $x, y \in \mathcal{L}$ such that $x \vee y \in F$ where $x \notin J(\mathcal{L})$. If $1 \neq z \in (1 :_{\mathcal{L}} x)$, then $z \vee x = 1$ gives $x \in \mathrm{I}(\mathcal{L}) \subseteq J(\mathcal{L})$, a contradiction. Thus, $(1 :_{\mathcal{L}} x) = \{1\}$. Since $F$ is an $r$-filter, we conclude that $y \in F$ and so $F$ is a $J$-filter.

(2) Let $x, y \in \mathcal{L}$ such that $x \vee y \in F$ with $x \notin J(\mathcal{L})$. By the hypothesis, there exist $c \in (1 :_{\mathcal{L}} a)$ and $d \in (1 :_{\mathcal{L}} b)$ such that $x \vee y = c \wedge d$. Since $(x \vee y) \vee (a \vee b) = (a \vee b) \vee (c \wedge d) = (a \vee b \vee c) \wedge (a \vee b \vee d) = 1$, we conclude that $y \vee a \vee b = 1$, as $x \notin \mathrm{I}(\mathcal{L})$. Consequently, $y \vee b \in (1 :_{\mathcal{L}} a)$ and $y \vee a \in (1 :_{\mathcal{L}} b)$. Therefore, $y = y \vee 0 = y \vee (a \wedge b) = (y \vee a) \wedge (y \vee b) \in F$ and so $F$ is a $J$-filter. □

**Proposition 3.8.** *If $F$ is a 1-filter of $\mathcal{L}$, then $F = \{1\}$.*

*Proof.* Assume that $F$ is a 1-filter but $\{1\} \subsetneqq F$. Then there exists an element $a \in F$ such that $a \neq 1$. Since $0 \vee a = a \in F$ and $F$ is a 1-filter, we conclude that $0 \in F$, so that $F = \mathcal{L}$, a contradiction. Hence $F = \{1\}$. □

In general, the class of $J$-filters is not comparable with the class of $r$-filters. Also, in the following example, it is shown that the condition "$\mathcal{L}$ is a presimplifiable lattice" in Proposition 3.7 cannot be omitted.

**Example 3.9.** (1) Let $\mathcal{L} = \{0, a, b, 1\}$ be a lattice with the relations $0 \leq a \leq b \leq 1$. Then nontrivial filters of $\mathcal{L}$ are $F = \{1, a, b\}$ and $G = \{1, b\}$. Therefore, $\mathcal{L}$ is a local lattice; so $G$ is a $J$-filter. On the other hand, $G$ is not a $r$-filter. For example, $a, b \in \mathcal{L}$ with $a \vee b = b \in G$ and $(1 :_{\mathcal{L}} b) = \{1\}$ but $a \notin G$.

(2) Let $\mathcal{L}$ be the lattice as in Example 3.5 (1). Clearly, $F = \{1\}$ is a $r$-filter of $\mathcal{L}$. On the other hand, $F$ is not a $J$-filter. For example, $\{a, b\}, \{c\} \in \mathcal{L}$ with $\{a, b\} \vee \{c\} \in F$ and $\{c\} \notin J(\mathcal{L})$ but $\{a, b\} \notin F$.

**Proposition 3.10.** *If $F$ is a $J$-filter of $\mathcal{L}$, then $F \ll \mathcal{L}$.*

*Proof.* Let $G$ be a filter of $\mathcal{L}$ such that $G \wedge F = \mathcal{L}$. Assume on the contrary, that $G \neq \mathcal{L}$. Since $G \neq \{1\}$, we conclude that $G \subseteq \mathbf{m}$ for some maximal filter $\mathbf{m}$ of $\mathcal{L}$ by [9, Lemma 2.1]. It follows from Proposition 3.2 that $F \subseteq J(\mathcal{L}) \subseteq \mathbf{m}$ and so $\mathcal{L} = F \wedge G \subseteq \mathbf{m}$ which is impossible. Thus $G = \mathcal{L}$, as needed. □

In the following theorem we give three other characterizations of $J$-filters.

**Theorem 3.11.** *If $F$ is a proper filter of a lattice $\mathcal{L}$, then the following statements are equivalent:*

*(1) $F$ is a $J$-filter of $\mathcal{L}$;*

*(2) $F = (F :_{\mathcal{L}} x)$ for every is $x \notin J(\mathcal{L})$;*

*(3) For filters $G$ and $K$ of $\mathcal{L}$, $G \vee K \subseteq F$ and $G \nsubseteq J(\mathcal{L})$ implies that $K \subseteq F$;*

*(4) $(F :_{\mathcal{L}} x) \subseteq J(\mathcal{L})$ for every $x \notin F$.*

*Proof.* (1) $\Rightarrow$ (2) Since the inclusion $F \subseteq (F :_{\mathcal{L}} x)$ for every $x \notin J(\mathcal{L})$ is clear, we will prove the reverse inclusion. Let $y \in (F :_{\mathcal{L}} x)$. Then $x \vee y \in F$ and $F$ is a $J$-filter gives $y \in F$ and so we have the equality.

(2) $\Rightarrow$ (3) Since $G \nsubseteq J(\mathcal{L})$, there exists $g \in G$ such that $g \notin J(\mathcal{L})$ which implies that $g \vee K \subseteq F$ and so $K \subseteq (F :_{\mathcal{L}} g) = F$ by (2).

(3) $\Rightarrow$ (1) Let $x, y \in \mathcal{L}$ with $x \vee y \in F$ and $x \notin J(\mathcal{L})$. Set $G = T(\{x\})$ and $K = T(\{y\})$. Then $G \vee K \subseteq F$ and $G \nsubseteq J(\mathcal{L})$ and so $y \in K \subseteq F$. Thus $F$ is a $J$-filter.

(1) $\Rightarrow$ (4) Assume that $x \notin F$ and let $y \in (F :_{\mathcal{L}} x)$. Then $x \vee y \in F$ and $F$ is a $J$-filter gives $y \in J(\mathcal{L})$, i.e. (4) holds.

(4) $\Rightarrow$ (1) Let $x, y \in \mathcal{L}$ such that $x \vee y \in F$ and $x \notin J(\mathcal{L})$. If $y \notin F$, then by (4), $x \in (F :_{\mathcal{L}} y) \subseteq J(\mathcal{L})$, a contradiction. Therefore, $y \in F$, i.e. (1) holds. □

**Lemma 3.12.** *Let $S$ be a nonempty subset of a lattice $\mathcal{L}$. If $F$ is a $J$-filter of $\mathcal{L}$ with $S \nsubseteq F$, then $(F :_{\mathcal{L}} S)$ is a $J$-filter of $\mathcal{L}$.*

*Proof.* Proof. If $(F :_{\mathcal{L}} S) = \mathcal{L}$, then $0 \in (F :_{\mathcal{L}} S)$ and so $S \subseteq F$ which is a contradiction. Thus, $(F :_{\mathcal{L}} S)$ is a proper filter of $\mathcal{L}$. Let $x \vee y \in (F :_{\mathcal{L}} S)$ and $x \notin J(\mathcal{L})$. Then $x \vee y \vee s \in F$ for every $s \in S$ and $F$ is a $J$-filter gives $y \vee s \in F$ and so $y \in (F :_{\mathcal{L}} S)$. □

A $J$-filter $F$ of $\mathcal{L}$ is called a *maximal $J$-filter* if there is no $J$-filter which contains $F$ properly.

**Proposition 3.13.** *Let $F$ be a proper filter of $\mathcal{L}$. Then the following hold:*

*(1) If $F$ is a prime filter of $\mathcal{L}$ with $F \subseteq J(\mathcal{L})$, then $F$ is a $J$-filter of $\mathcal{L}$.*

*(2) If $F$ is a maximal $J$-filter of $\mathcal{L}$, then $F$ is a prime filter.*

*(3) $J(R)$ is a prime filter if and only if $J(R)$ is a maximal $J$-filter of $\mathcal{L}$.*

*Proof.* (1) Let $x, y \in \mathcal{L}$ with $x \vee y \in F$ and $x \notin J(\mathcal{L})$ (so $x \notin F$). Then $F$ is prime gives $y \in F$. So $F$ is a $J$-filter.

(2) Let $x, y \in \mathcal{L}$ such that $x \vee y \in F$ and $x \notin F$. Then $(F :_{\mathcal{L}} \{x\})$ is a $J$-filter by Lemma 3.12 and $F \subseteq (F :_{\mathcal{L}} \{x\})$. By maximality of $F$, we have $y \in (F :_{\mathcal{L}} \{x\}) = F$ and so $F$ is a prime filter of $\mathcal{L}$.

(3) Let $J(\mathcal{L})$ be a prime filter of $\mathcal{L}$. Then by (1), it is a $J$-filter. Moreover, $J(\mathcal{L})$ is maximal by Proposition 3.2. The converse is a consequence of (2). □

A lattice $\mathcal{L}$ is called *semiprimitive* if $J(\mathcal{L}) = \{1\}$. The following example shows that any lattice $\mathcal{L}$ need not have an $J$-filter.

**Example 3.14.** Let $D = \{a, b\}$. Then $\mathcal{L} = \{X : X \subseteq D\}$ forms a distributive lattice under set inclusion with greatest element $D$ and least element $\emptyset$. It can be easily seen that the set of all proper filters of $\mathcal{L}$ is $\{\{D\}, F_1, F_2\}$, where $F_1 = \{D, \{a\}\}$, $F_2 = \{D, \{b\}\}$ and $J(\mathcal{L}) = \{1\}$ (so $\mathcal{L}$ is semiprimitive). Also, it is not a $\mathcal{L}$-domain since $\{a\} \vee \{b\} = 1$ but $\{a\} \neq 1$ and $\{b\} \neq 1$. Clearly, $\mathcal{L}$ has no $J$-filters.

**Proposition 3.15.** *If $\mathcal{L}$ is a semiprimitive lattice which is not a $\mathcal{L}$-domain, then $\mathcal{L}$ has no $J$-filters.*

*Proof.* On the contrary, assume that $F$ is a $J$-filter of $\mathcal{L}$; so $F = \{1\}$ by Proposition 3.2. By the hypothesis, there exist $a, b \in \mathcal{L}$ such that $a \vee b = 1$, $1 \neq a \notin J(R) = \{1\}$ and $1 \neq b \notin F$, a contradiction. Thus $\mathcal{L}$ has no $J$-filters. □

**Theorem 3.16.** *Let $\mathcal{L}$ be a semiprimitive lattice. Then the following statements are equivalent:*

*(1) $\mathcal{L}$ is a $\mathcal{L}$-domain;*

*(2) The filter $\{1\}$ is the only $J$-filter of $\mathcal{L}$;*

*(3) $(1 :_{\mathcal{L}} x \vee y) = (1 :_{\mathcal{L}} x) \cup (1 :_{\mathcal{L}} y)$ for every $x, y \in \mathcal{L}$.*

*Proof.* (1) $\Rightarrow$ (2) If $F$ is a $J$-filter of $\mathcal{L}$, then $F = \{1\}$ by Proposition 3.2, as $J(\mathcal{L}) = \{1\}$. Assume that $\mathcal{L}$ is a $\mathcal{L}$-domain and let $x, y \in \mathcal{L}$ such that $x \vee y \in \{1\} = F$ and $x \notin J(\mathcal{L})$. Then $\mathcal{L}$ is a $\mathcal{L}$-domain gives $y \in F$. Thus $F$ is a $J$-filter.

(2) $\Rightarrow$ (3) At first, we show that $(1 :_{\mathcal{L}} z)$ is a $J$-filter for each $1 \neq z \in \mathcal{L}$. Let $a, b \in \mathcal{L}$ such that $a \vee b \in (1 :_{\mathcal{L}} z)$ and $a \notin J(\mathcal{L})$. Then $\{1\}$ is a $J$-filter gives $b \vee z = 1$; hence $b \in (1 :_{\mathcal{L}} z)$. Therefore, by our hypothesis, we have $(1 :_{\mathcal{L}} z) = \{1\}$ for each $1 \neq z \in \mathcal{L}$. This immediately implies that $(1 :_{\mathcal{L}} x \vee y) = (1 :_{\mathcal{L}} x) \cup (1 :_{\mathcal{L}} y)$ for every $x, y \in \mathcal{L}$.

(3) $\Rightarrow$ (1) Let $x \vee y = 1$, where $x, y \in \mathcal{L}$. Then $\mathcal{L} = (1 :_{\mathcal{L}} x \vee y) = (1 :_{\mathcal{L}} x) \cup (1 :_{\mathcal{L}} y)$ implies that $0 \in (1 :_{\mathcal{L}} x) \cup (1 :_{\mathcal{L}} y)$. This means that $x = 1$ or $y = 1$, i.e. $\mathcal{L}$ is a $\mathcal{L}$-domain. □

For a filter $F$ of a lattice $\mathcal{L}$, the Jacobson radical of $F$, denoted by $J(F)$, is defined as the intersection of all maximal filters of $\mathcal{L}$ containing $F$. The following lemma can be easily proved for any filters $F$ and $G$ of $\mathcal{L}$:

**Lemma 3.17.** *Let $F, G$ be filters of $\mathcal{L}$. Then the following hold:*
*(1) $F \subseteq J(F)$.*
*(2) If $F \subseteq G$, then $J(F) \subseteq J(G)$.*
*(3) $J(\mathcal{L}) \subseteq J(F)$.*
*(4) $J(J(F)) = J(F)$.*

**Definition 3.18.** A proper filter $F$ of a lattice $\mathcal{L}$ is called *$J$-primary* if whenever $x, y \in \mathcal{L}$, $x \vee y \in F$ implies that $x \in J(F)$ or $y \in F$.

In the following result, we show that $J$-filters and $J$-primary filters that contained in $J(\mathcal{L})$ are the same.

**Proposition 3.19.** *If $F$ is a proper filter of a lattice $\mathcal{L}$ such that $F \subseteq J(\mathcal{L})$, then $F$ is a $J$-filter if and only if $F$ is $J$-primary.*

*Proof.* Suppose $F$ is a $J$-filter of $\mathcal{L}$ and let $x \vee y \in F$ with $x \notin J(F)$. Since $J(\mathcal{L}) \subseteq J(F)$, we conclude that$x \notin J(\mathcal{L})$. Now, $F$ is a $J$-filter gives $y \in F$ and so $F$ is a $J$-primary filter of $\mathcal{L}$. Conversely, assume that $F$ is a $J$-primary filter of $\mathcal{L}$ and let $a \vee b \in F$ with $a \notin J(\mathcal{L})$. Now, $F \subseteq J(\mathcal{L})$ implies that $J(F) \subseteq J(J(\mathcal{L}) = J(\mathcal{L})$. Therefore, $a \notin J(F)$ and so $b \in F$, as $F$ is $J$-primary. Hence, $F$ is a $J$-filter. □

**Proposition 3.20.** *Let $F$ be a filter of a lattice $\mathcal{L}$ such that $F \nsubseteq J(\mathcal{L})$. Then the following hold:*
*(1) If $G$ and $K$ are $J$-filters of $\mathcal{L}$ with $G \vee F = K \vee F$, then $G = K$.*
*(2) If $G$ is a filter such that $G \vee F$ is a $J$-filter, then $G \vee F = G$.*

*Proof.* (1) Clearly, $G \vee F = K \vee F \subseteq G$ and $G \vee F = K \vee F \subseteq K$. Since $G$ is a $J$-filter and $F \nsubseteq J(\mathcal{L})$, we conclude that $K \subseteq G$ by Theorem 3.11. Similarly, as $K$ is a $J$-filter, we obtain $G \subseteq K$, and so we have equality.

(2) Since the inclusion $G \vee F \subseteq G$ is clear, we will prove the reverse inclusion. By the hypothesis, $G \vee F \subseteq G \vee F$ and $F \nsubseteq J(\mathcal{L})$ gives $G \subseteq G \vee F$ by Theorem 3.11, as required. □

**Proposition 3.21.** *Let $\mathcal{L}$ be a lattice. Then the following hold:*
*(1) If $\{F_i\}_{i \in \Lambda}$ is a nonempty family of $J$-filters of $\mathcal{L}$, then $\cap_{i \in \Lambda} F_i$ is a $J$-filter.*

*(2) Let $P_1, \cdots, P_n$ be prime filters of $\mathcal{L}$ which are not comparable. If $\cap_{i=1}^n P_i$ is a $J$-filter, then $P_i$ is a $J$-filter for $i \in \{1, \cdots, n\}$.*

*Proof.* (1) Let $x, y \in \mathcal{L}$ such that $x \vee y \in \cap_{i\in\Lambda} F_i$ and $x \notin J(\mathcal{L})$. By the hypothesis, $x \vee y \in F_i$ for all $i \in \Lambda$ gives $y \in F_i$ for all $i \in \Lambda$ and so $y \in \cap_{i\in\Lambda} F_i$, as needed.

(2) Let $x \vee y \in P_i$ with $x \notin J(\mathcal{L})$ and take $z \in (\bigvee_{i\neq j} P_j) \setminus P_i$. Therefore, $x \vee y \vee z \in \bigvee_{i=1}^n P_i = \cap_{i=1}^n P_i$. Since $\cap_{i=1}^n P_i$ is a $J$-filter and $x \notin J(\mathcal{L})$, we conclude that $y \vee z \in \cap_{i=1}^n P_i$ and so $y \vee z \in P_i$. This implies that $y \in P_i$, i.e. $P_i$ is a $J$-filter of $\mathcal{L}$. □

If $F$ is a filter of $\mathcal{L}$, the the set of all prime filters of $\mathcal{L}$ that are minimal over $F$ is denoted by $\min(F)$.

**Theorem 3.22.** *Let $\mathcal{L}$ be a lattice and $P \in \min(F)$, where $F$ is a $J$-filter of $\mathcal{L}$. Then $P$ is a $J$-filter.*

*Proof.* Let $x, y \in \mathcal{L}$ such that $x \vee y \in P$ with $x \notin J(\mathcal{L})$. By [6, Proposition 2.7], there exists $z \notin P$ such that $x \vee y \vee z \in F$. Since $x \notin J(\mathcal{L})$ and $F$ is a $J$-filter, we conclude that $y \vee z \in F \subseteq P$ which implies that $y \in P$, as $P$ is a prime filter. Therefore, $P$ is a $J$-filter. □

**Lemma 3.23.** *Let $F$ be a filter of a lattice $\mathcal{L}$. Then $F$ is a $J$-filter if and only if whenever $G$ and $K$ are filters of $\mathcal{L}$ with $G \cap (\mathcal{L} \setminus J(\mathcal{L}) \neq \emptyset$ and $G \vee K \subseteq F$, then $K \subseteq F$.*

*Proof.* Assume that $F$ is a $J$-filter and let $k \in K$. By assumption, there exists $g \in G$ such that $g \notin J(\mathcal{L})$; so $g \vee k \in G \vee K \subseteq F$ gives $k \in F$, as $F$ is a $J$-filter. Thus $K \subseteq F$. Conversely, let $x, y \in \mathcal{L}$ such that $x \vee y \in F$ with $x \notin J(\mathcal{L})$. Set $G = T(\{x\})$ and $K = T(\{y\})$. Thus, $G \vee K \subseteq F$ and $x \in G \cap (\mathcal{L} \setminus J(\mathcal{L})$. Then $y \in K \subseteq F$, as required. □

Let $G \subseteq \cup_{i\in I} F_i$, where $G$, $F_i$s are subsets of a lattice $\mathcal{L}$. This inclusion is called *irreducible* if no $F_i$ can be removed from the union [12].

The following theorem is a lattice counterpart of the Prime Avoidance Lemma for $J$-filters. Compare the next Theorem with Theorem 3.8 in [12].

**Theorem 3.24.** *Let $G \subseteq \cup_{i=1}^n F_i$, where $G, F_1, \cdots, F_n$ are filters of a lattice $\mathcal{L}$, be an irreducible inclusion. If $F_1$ is a $J$-filter and $F_i \cap (\mathcal{L} \setminus J(\mathcal{L})) \neq \emptyset$, for all $i \geq 2$, then $G \subseteq F_1$.*

*Proof.* Since $G \nsubseteq \cup_{i=2}^n F_i$, we conclude that there exists $g \in G$ such that $g \notin \cup_{i=2}^n F_i$. This shows that $g \in F_1$. If $e \in G \cap (\cup_{i=2}^n F_i)$, then $e \wedge g \notin \cup_{i=2}^n F_i$ (otherwise, $e \wedge g \in F_j$ for some $2 \leq j \leq n$; hence $g \in F_j$ by Lemma 2.1 (1), a contradiction). As $g \wedge e \in G \subseteq \cup_{i=1}^n F_i$, we infer that $g \wedge e \in F_1$ which implies that $e \in F_1$ by lemma 2.1 (1). This shows that $G \cap (\cup_{i=2}^n F_i \subseteq F_1$ and so $G \vee (\bigvee_{i=2}^n F_i) \subseteq F_1$. Since $(\bigvee_{i=2}^n F_i) \cap ((\mathcal{L} \setminus J(\mathcal{L})) \neq \emptyset$, by Lemma 3.23, we infer that $G \subseteq F_1$. □

## 4. Further Results

We begin this section with the investigation of the stability of $J$-filters in various lattice-theoretic constructions.

**Lemma 4.1.** *If $f : \mathcal{L} \to \mathcal{L}'$ is a lattice epimorphism such that $f(0) = 0$, then the following hold:*

*(1) If* $\mathbf{m}$ *is a maximal filter of* $\mathcal{L}$*, then* $f(\mathbf{m}) \neq \{1\}$ *is maximal in* $\mathcal{L}'$*.*
*(2)* $f(J(\mathcal{L}) \subseteq J(\mathcal{L}')$*.*

*Proof.* (1) Let $f(\mathbf{m}) \subsetneqq H \subseteq \mathcal{L}'$ for some filter $H$ of $\mathcal{L}'$. Then there exists $x \in \mathcal{L} \setminus \mathbf{m}$ such that $f(x) \in H \setminus f(\mathbf{m})$. Since $\mathbf{m} \subsetneqq \mathbf{m} \wedge T(\{x\})$, we conclude that $\mathbf{m} \wedge T(\{x\}) = \mathcal{L}$ and so $0 = m \wedge (x \vee a) = (m \wedge x) \vee (m \wedge a)$ for some $m \in \mathbf{m}$ and $a \in \mathcal{L}$; hence $m \wedge x = 0$. It follows that $f(m \wedge x) = f(m) \wedge f(x) = f(0) = 0 \in H$ which implies that $H = \mathcal{L}'$. Thus $f(\mathbf{m})$ is a maximal filter of $\mathcal{L}'$.

(2) We have $f(J(\mathcal{L}) = f(\bigcap_{\mathbf{m} \in \mathrm{Max}(\mathcal{L})} \mathbf{m}) \subseteq \bigcap_{\mathbf{m} \in \mathrm{Max}(\mathcal{L})} f(\mathbf{m}) \subseteq J(\mathcal{L}')$. □

**Theorem 4.2.** *Let* $\mathcal{L}$ *be a complemented lattice. If* $f : \mathcal{L} \to \mathcal{L}'$ *is a lattice epimorphism such that* $f(0) = 0$*, then the following hold:*
*(1) If* $G$ *is a* $J$*-filter of* $\mathcal{L}$ *with* $\ker(\mathrm{f}) \subseteq G$*, then* $f(G)$ *is a* $J$*-filter of* $\mathcal{L}'$*.*
*(2) If* $K$ *is a* $J$*-filter of* $\mathcal{L}'$ *with* $\ker(\mathrm{f}) \subseteq J(\mathcal{L})$*, then* $f^{-1}(K)$ *is a* $J$*-filter of* $\mathcal{L}$*.*

*Proof.* (1) Let $x, y \in \mathcal{L}'$ such that $x \vee y \in f(G)$ and $x \notin J(\mathcal{L}')$. Since f is an epimorphism, there exist $a, b \in \mathcal{L}$ such that $x = f(a)$ and $y = f(b)$. Then $x \vee y = f(a \vee b) \in f(G)$ and so $f(a \vee b) = f(g)$ for some $g \in G$. By the hypothesis, $g \vee g' = 1$ and $g \wedge g' = 0$ for some $g' \in \mathcal{L}$. Since $f(a \vee b \vee g') = f(a \vee b) \vee f(g') = 1$, we conclude that $a \vee b \vee g' \in \ker(f) \subseteq G$; hence $a \vee b = (a \vee b) \vee (g \wedge g') = (a \vee b \vee g) \wedge (a \vee b \vee g') \in G$, as $G$ is a filter. Also, note that $a \notin J(\mathcal{L})$ since otherwise if $a \in J(\mathcal{L})$, then $x = f(a) \in J(\mathcal{L}')$ by Lemma 4.1 which is impossible. As $G$ is a $J$-filter, we get that $b \in G$ and so $y = f(b) \in f(G)$, as needed.

(2) Let $a, b \in \mathcal{L}$ with $a \vee b \in f^{-1}(K)$ and $a \notin J(\mathcal{L})$. Then $f(a) \vee f(b) = f(a \vee b) \in K$. We show that $f(a) \notin J(\mathcal{L}')$. On the contrary, assume that $f(a) \in J(\mathcal{L}')$. Let $\mathbf{m}$ be a maximal filter of $\mathcal{L}$. Then $f(\mathbf{m})$ is a maximal filter of $\mathcal{L}'$ since $\ker(f) \subseteq J(\mathcal{L}) \subseteq \mathbf{m}$. Now, since $f(a) \in J(\mathcal{L}') \subseteq f(\mathbf{m})$, we conclude that $f(a) = f(m)$ for some $m \in \mathbf{m}$. By assumption, $m \vee m' = 1$ and $m \wedge m' = 0$ for some $m' \in \mathcal{L}$. As $f(a \vee m') = f(m) \vee f(m') = 1$, we conclude that $a \vee m' \in \ker(f) \subseteq \mathbf{m}$; hence $a = a \vee (m \wedge m') = (a \vee m) \wedge (a \vee m') \in \mathbf{m}$, as $\mathbf{m}$ is a filter. Therefore, $a \in J(\mathcal{L})$ which is a contradiction. Since $K$ is a J-filter, $f(b) \in K$ and so $b \in f^{-1}(K)$, as needed. □

Quotient lattices are determined by equivalence relations rather than by ideals as in the ring case. If $F$ is a filter of a lattice $(\mathcal{L}, \leq)$, we define a relation on $\mathcal{L}$, given by $x \sim y$ if and only if there exist $a, b \in F$ satisfying $x \wedge a = y \wedge b$. Then $\sim$ is an equivalence relation on $\mathcal{L}$, and we denote the equivalence class of $a$ by $a \wedge F$ and these collection of all equivalence classes by $\mathcal{L}/F$. We set up a partial order $\leq_Q$ on $\mathcal{L}/F$ as follows: for each $a \wedge F, b \wedge F \in \mathcal{L}/F$, we write $a \wedge F \leq_Q b \wedge F$ if and only if $a \leq b$. The following notation below will be used in this paper: It is straightforward to check that $(\mathcal{L}/F, \leq_Q)$ is a lattice with $(a \wedge F) \vee_Q (b \wedge F) = (a \vee b) \wedge F$ and $(a \wedge F) \wedge_Q (b \wedge F) = (a \wedge b) \wedge F$ for all elements $a \wedge F, b \wedge F \in \mathcal{L}/F$. Note that $f \wedge F = F$ if and only if $f \in F$ (see [8, Remark 4.2 and Lemma 4.3]).

**Lemma 4.3.** *If* $F$ *is a filter of* $\mathcal{L}$*, then* $J(\mathcal{L}/F) = J(F)/F$*.*

*Proof.* Let $\mathbf{m}/F \in \mathrm{Max}(\mathcal{L}/F)$. If $\mathbf{m} \subsetneqq P \subseteq \mathcal{L}$ for some filter $P$ of $\mathcal{L}$, then $\mathbf{m}/F \subsetneqq P/F \subseteq \mathcal{L}/F$ gives $P/F = \mathcal{L}/F$; so $P = \mathcal{L}$ by [8, Lemma 4.3]. Thus $\mathbf{m} \in \mathrm{Max}(\mathcal{L})$ with $F \subseteq \mathbf{m}$. Similarly, if $P \in \mathrm{Max}(\mathcal{L})$ with $F \subseteq P$, then $P/F \in \mathrm{Max}(\mathcal{L}/F)$, as required. □

**Proposition 4.4.** *Suppose* $\mathcal{L}$ *is a complemented lattice and let* $F, G$ *be two filters of* $\mathcal{L}$ *with* $F \subseteq G$*. Then the followings hold:*

*(1) If $G$ is a $J$-filter of $\mathcal{L}$, then $G/F$ is a $J$-filter of $\mathcal{L}/F$.*
*(2) If $G/F$ is a $J$-filter of $\mathcal{L}/\mathcal{F}$ and $F \subseteq J(\mathcal{L})$, then $G$ is a $J$-filter of $\mathcal{L}$.*
*(3) If $G/F$ is a $J$-filter of $\mathcal{L}/F$ and $F$ is a $J$-filter of $\mathcal{L}$, then $G$ is a $J$-filter of $\mathcal{L}$.*

*Proof.* (1) Let $v : \mathcal{L} \to \mathcal{L}/F$ be the natural epimorphism defined by $v(x) = x \wedge F$. Then $\ker(v) = \{x \in \mathcal{L} : x \wedge F = 1 \wedge F\} = F \subseteq G$ by [8, Lemma 4.3] and so by Theorem 4.2 (1) and [8, Lemma 4.3] that $v(G) = \{x \wedge F : x \in G\} = G/F$ is a $J$-filter of $\mathcal{L}/F$.

(2) Consider the natural epimorphism $v : \mathcal{L} \to \mathcal{L}/F$ defined by $v(x) = x \wedge F$. Since $F \subseteq J(\mathcal{L})$, we conclude that $v^{-1}(G/F) = G$ is a $J$-filter of $\mathcal{L}$ by Theorem 4.2 (2).

(3) Since $F$ is a $J$-filter of $\mathcal{L}$, we conclude that $F \subseteq J(\mathcal{L})$ by Proposition 3.2. Now the assertion follows from (2). □

If $F$ is a proper filter of $\mathcal{L}$, then by a $\vee$-*factorization* of $F$ we mean an expression of $F$ as an intersection $\bigcap_{i=1}^{n} F_i = \bigvee_{i=1}^{n} F_i$ of $J$-filters. We call $\mathcal{L}$ a $\vee$-*lattice* if every proper filter has a $\vee$-factorization.

**Theorem 4.5.** *Let $F$ be a proper filter of a complemented lattice $\mathcal{L}$. If $\mathcal{L}$ is a $\vee$-lattice, then $\mathcal{L}/F$ is a $\vee$-lattice.*

*Proof.* Let $H$ be a proper filter of $\mathcal{L}/F$. Then $H = K/F$ for some proper filter $K$ of $\mathcal{L}$ by [8, Lemma 4.2 (5)]. Let $K = \bigcap_{i=1}^{n} F_i$ be a $\vee$-factorization of $K$. Then $H = (\bigcap_{i=1}^{n} F_i)/F = \bigcap_{i=1}^{n} F_i/F$ by [8, Lemma 4.2 (7)]. By Proposition 4.4 (1), we conclude that $F_i/F$ is a $J$-filter of $\mathcal{L}/F$ for each $i \in \{1, 2, \cdots, n\}$. It means that $\mathcal{L}/F$ is a $\vee$-lattice. □

**Definition 4.6.** If $S$ is a nonempty subset of a lattice $\mathcal{L}$ such that $\mathcal{L} \setminus J(\mathcal{L}) \subseteq S$, then $S$ is called a $J$-*join-subset* of $\mathcal{L}$ if $x \vee y \in S$ for all $x \in \mathcal{L} \setminus J(\mathcal{L})$ and all $y \in S$.

In the following theorem, we describe the relation between $J$-filters and $J$-join-subsets of $\mathcal{L}$.

**Proposition 4.7.** *If $F$ is a proper filter of $\mathcal{L}$, then the following statements are equivalent:*
*(1) $F$ is a $J$-filter of $\mathcal{L}$;*
*(2) $\mathcal{L} \setminus F$ is a $J$-join-subset of $\mathcal{L}$.*

*Proof.* (1) $\Rightarrow$ (2) Since $F \subseteq J(\mathcal{L})$ by Proposition 3.2, we conclude that $\mathcal{L} \setminus J(\mathcal{L}) \subseteq \mathcal{L} \setminus F$. Let $x \in \mathcal{L} \setminus J(\mathcal{L})$ and $y \in \mathcal{L} \setminus F$. We show that $x \vee y \in \mathcal{L} \setminus F$. On the contrary, assume that $x \vee y \in F$. Then $F$ is a $J$-filter and $x \notin J(\mathcal{L})$ implies that $y \in F$ which is a contradiction. Thus, $x \vee y \in \mathcal{L} \setminus F$ and so $\mathcal{L} \setminus F$ is a $J$-join-subset of $\mathcal{L}$.

(2) $\Rightarrow$ (1) Let $a, b \in \mathcal{L}$ and $a \vee b \in F$ with $a \notin J(\mathcal{L})$. Then we have $b \in F$ since otherwise we would have $a \vee b \in \mathcal{L} \setminus F$ which is impossible. Therefore, $F$ is a $J$-filter of $\mathcal{L}$. □

Let $\mathbb{F}(\mathcal{L})$ be the set of all filters of $\mathcal{L}$. The following proposition is a lattice counterpart of Theorem 3.44 in [13] describing the structure of $J$-filters.

**Proposition 4.8.** *Suppose $F$ is a filter of $\mathcal{L}$ and let $S$ be a $J$-join-subset of $\mathcal{L}$ with $F \cap S = \emptyset$. Then there exists a $J$-filter $G$ containing $F$ such that $G \cap S = \emptyset$.*

*Proof.* Set $\Omega = \{K \in \mathbb{F}(\mathcal{L}) : F \subseteq K, K \cap S = \emptyset\}$. Since $F \in \Omega$, we conclude that $\Omega \neq \emptyset$. Clearly, $(\Omega, \subseteq)$ is a partial order. Let $\{F_i\}_{i \in \Lambda}$ be a chain of elements of $\Omega$ and set $G = \cup_{i \in \Lambda} F_i$. It is clear that $G$ is a filter of $\mathcal{L}$ and $G \in \Omega$. Thus $G$ is an upper bound of the chain $\{F_i\}_{i \in \Lambda}$; hence by Zorn's Lemma, $\Omega$ has a maximal element $G$. It remains to show that $G$ is a $J$-filter of

$\mathcal{L}$. On the contrary, assume that $x \vee y \in G$ for some $x \notin J(\mathcal{L})$ and $y \notin G$. Since $G \subsetneqq (G :_{\mathcal{L}} x)$, we get $(G :_{\mathcal{L}} x) \cap S \neq \emptyset$ by maximality of $G$ and so there exists $s \in S$ such that $s \vee x \in G \cap S$ which is a contradiction. Therefore, $G$ is a $J$-filter of $\mathcal{L}$ with $G \cap S = \emptyset$ and $F \subseteq G$. □

We denote by $\mathrm{JSpec}(\mathcal{L})$ the set of all $J$-filters of $\mathcal{L}$. If $F$ is a filter in $\mathcal{L}$, the set of all *minimal* $J$-filters over $F$ will be denoted by $\min_J(F)$ (set $\min_J(\{1\}) = \min_J(\mathcal{L})$) and $V_J(F) = \{H \in \mathrm{JSpec}(\mathcal{L}) : F \subseteq H\}$. Compare the next Theorem with Theorem 2.1, p. 2 in [10].

**Theorem 4.9.** *Let $F \subseteq P$ be filters of $\mathcal{L}$ with $P$ is a $J$-filter. Then the following statements are equivalent:*

*(1) $P \in \min_J(F)$;*

*(2) $\mathcal{L} \setminus P$ is a $J$-join-subset that is maximal with $(\mathcal{L} \setminus P) \cap F = \emptyset$;*

*(3) For each $x \in P$, there is a $y \notin P$ such that $x \vee y \in F$.*

*Proof.* (1) ⇒ (2) Since $(\mathcal{L} \setminus P) \cap F = \emptyset$, the set $\Omega$ of all $J$-join-subsets, say $H$, with $H \cap F = \emptyset$ is not empty. Of course, the relation of inclusion, $\subseteq$, is a partial order on $\Omega$. Now $\Omega$ is easily seen to be inductive under inclusion, so by Zorn's Lemma $\Omega$ has a maximal element $S$ (so $F \cap S = \emptyset$). By Proposition 4.8, there is a filter $Q$ of $\mathcal{L}$ containing $F$ that is maximal with respect to being disjoint from $S$ which is $J$-filter. Note that $Q \cap (\mathcal{L} \setminus P) = \emptyset$ which implies that $P = Q$. Hence, $S = \mathcal{L} \setminus P$.

(2) ⇒ (3) Let $1 \neq x \in P$ and set $S = \{y \vee (\wedge_{j=1}^{n} x) : y \in \mathcal{L} \setminus P, n = 0, 1, \cdots\}$ (note that $\wedge_{i=1}^{0} x$ is interpreted as 0, and clearly, $\wedge_{i=1}^{n} x = x$). Since $P$ is a $J$-filter of $\mathcal{L}$, we conclude that $\mathcal{L} \setminus P$ is a $J$-join-subset of $\mathcal{L}$ by Proposition 4.7; so $\mathcal{L} \setminus J(\mathcal{L}) \subseteq \mathcal{L} \setminus P \subseteq S$. Let $x \vee z \in S$ for some $z \in \mathcal{L} \setminus P$ and $u \in \mathcal{L} \setminus J(\mathcal{L})$. Then $u \vee z \notin P$ (otherwise, $u \notin J(\mathcal{L})$ and $u \vee z \in P$ gives $z \in P$, a contradiction) and so $(x \vee z) \vee u \in S$. Then S is a $J$-join-subset that properly contains $\mathcal{L} \setminus P$; so $F \cap S \neq \emptyset$ by maximality of $\mathcal{L} \setminus P$. Thus there exists $y \in \mathcal{L} \setminus P$ such that $x \vee y \in F$.

(3) ⇒ (1) Let $Q$ be a $J$-filter such that $F \subsetneqq Q \subseteq P$. If $P \neq Q$, then there is an element $x \in P$ with $x \notin Q$; so $x \vee y \in F \subsetneqq Q$ for some $y \notin P$ which is a contradiction. Therefore $P = Q$. □

**Corollary 4.10.** *Let $P$ be a $J$-filter of $\mathcal{L}$. Then the following statements are equivalent:*

*(1) $P \in \min_J(\mathcal{L})$;*

*(2) $\mathcal{L} \setminus P$ is a $J$-join-subset that is maximal with $(\mathcal{L} \setminus P) \cap F = \emptyset$;*

*(3) For each $x \in P$, there is a $y \notin P$ such that $x \vee y = 1$.*

*Proof.* Take $F = \{1\}$ in Theorem 4.9. □

**Theorem 4.11.** *Let $P$ be a prime filter of a presimplifiable lattice $\mathcal{L}$. If $P \in \min_J(\mathcal{L})$ and $e \in \mathcal{L}$ has a complement in $\mathcal{L}$, then $F = P \wedge (1 :_{\mathcal{L}} e)$ is a $J$-filter of $\mathcal{L}$.*

*Proof.* Let $x, y \in \mathcal{L}$ such that $x \vee y \in F$ with $x \notin J(\mathcal{L})$. Hence, $x \vee y = a \wedge b$, where $a \in P$ and $b \vee e = 1$. By Corollary 4.10, there exists $c \notin P$ such that $a \vee c = 1$. Therefore, $(e \vee c) \vee (x \vee y) = (e \vee c) \vee (a \wedge b) = (e \vee c \vee a) \wedge (e \vee c \vee b) = 1$. Since $x \notin J(\mathcal{L})$, we conclude that $x \notin \mathrm{I}(\mathcal{L})$; so $e \vee c \vee y = 1 \in P$ which implies that $e \vee y \in P$, as $P$ is prime. By the hypothesis, there is an element $e' \in \mathcal{L}$ such that $e \vee e' = 1$ and $e \wedge e' = 0$. Now $y = y \vee (e \wedge e') = (e \vee y) \wedge (e' \vee y) \in P \wedge (1 :_{\mathcal{L}} e) = F$, and therefore $F$ is a $J$-filter. □

**Theorem 4.12.** *If $F$ is a filter of $\mathcal{L}$, then $F = \bigcap_{G \in V_J(F)} G$.*

*Proof.* Since the inclusion $F \subseteq \bigcap_{G\in V_J(F)} G$ is clear, we will prove the reverse inclusion. Let $x \in \bigcap_{G\in V_J(F)} G$. We suppose that $x \notin F$, and look for a contradiction. Set $S = \bigcap_{G\in V_J(F)}(\mathcal{L}\backslash G)$. One can easily show that $S$ is a $J$-join-subset of $\mathcal{L}$ with $S \cap F = \emptyset$ by Proposition 4.7. Set

$$\Omega = \{K \in \mathbb{F}(\mathcal{L}) : F \subseteq K, K \cap S = \emptyset, x \notin K\}.$$

Since $F \in \Omega$, we conclude that $\Omega \neq \emptyset$. By an argument like that in Proposition 4.8, there exists a $J$-filter $K$ containing $F$ such that $K \cap S = \emptyset$ and $x \notin K$. It follows that $K \in V_J(F)$, so that $x \in K$, a contradiction, i.e. we have equality. □

**Corollary 4.13.** $\bigcap_{G\in \mathrm{JSpec}(\mathcal{L})} G = \{1\}$.

*Proof.* Take $F = \{1\}$ in Theorem 4.12. □

**Proposition 4.14.** *If $F, G$ are filters with $G$ a $J$-filter and $F \subseteq G$, then there exists a minimal $J$-filter $K$ over $F$ such that $K \subseteq G$.*

*Proof.* Set $\Omega = \{H \in \mathrm{JSpec}(\mathcal{L}) : F \subseteq H \subseteq G\}$. Then $G \in \Omega$, and so $\Omega \neq \emptyset$. Put $S = \mathcal{L} \setminus G$. Then $S$ is a $J$-join-subset of $\mathcal{L}$ by Proposition 4.7. By an argument like that in Proposition 4.8, the set $\Omega$ of $J$-filters of $\mathcal{L}$ has a minimal member with respect to inclusion (by partially ordering $\Omega$ by reverse inclusion and using Zorn's Lemma) which is a $J$-filter, i.e. the result holds. □

**Theorem 4.15.** *Let $F$ be a proper filter of $\mathcal{L}$. Then the following hold:*

*(1) $F = \bigcap_{G\in \min_J(F)} G$;*

*(2) If $F$ is a $J$-filter, then $|\min_J(F)| = 1$;*

*(3) If $F$ has a $\vee$-factorization, then $\min_J(F)$ is finite.*

*Proof.* (1) Since $\min_J(F) \subseteq V_J(F)$, it is clear that $\bigcap_{G\in V_J(F)} G$ is a subset of $\bigcap_{G\in \min_J(F)} G$ by Theorem 4.12. However, the reverse inclusion is immediate from Proposition 4.14 (since every $J$-filter in $V_J(F)$ contains a minimal prime $J$-filter over $F$).

(2) This follows from (1).

(3) If $F = \bigcap_{i=1}^{k} F_i$ is a $\vee$-factorization, then $\min_J(F) \subseteq \bigcup_{i=1}^{k} \min_J(F_i)$, and so $|\min_J(F)| \leq k$ by (2), i.e. (3) holds. □

Assume that $(\mathcal{L})_1, \leq_1), (\mathcal{L})_2, \leq_2)$ are lattices and let $\mathcal{L} = \mathcal{L}_1 \times \mathcal{L}_2$. We set up a partial order $\leq_c$ on $\mathcal{L}$ as follows: for each $x = (x_1, x_2), y = (y_1, y_2) \in \mathcal{L}$, we write $x \leq_c y$ if and only if $x_i \leq_i y_i$ for each $i \in \{1, 2\}$. The following notation below will be used in this paper: It is straightforward to check that $(\mathcal{L}, \leq_c)$ is a lattice with $x \vee_c y = (x_1 \vee y_1, x_2 \vee y_2)$ and $x \wedge_c y = (x_1 \wedge y_1, x_2 \wedge y_2)$. In this case, we say that $\mathcal{L}$ is a *decomposable lattice*.

**Lemma 4.16.** *Let $\mathcal{L} = \mathcal{L}_1 \times \mathcal{L}_2$ be a decomposable lattice. If $P$ is a maximal filter of $\mathcal{L}_1$ and $Q$ is a maximal filter of $\mathcal{L}_2$, then $\mathbf{m} = P \times \mathcal{L}_2$ and $\mathbf{m}' = \mathcal{L}_1 \times Q$ are maximal filters of $\mathcal{L}$. In particular, $(1, 0), (0, 1) \notin J(\mathcal{L})$.*

*Proof.* On the contrary, assume that $\mathbf{m} \subsetneqq G \subsetneqq \mathcal{L}$ for some filter $G$ of $\mathcal{L}$. Then $G$ is of the form $K_1 \times \mathcal{L}_2$ such that $P \subsetneqq K_1 \subsetneqq \mathcal{L}_1$. There exists $x \in K_1$ such that $x \notin P$. Then $T(\{x\}) \wedge P = \mathcal{L}_1$ gives $0 = p \wedge (x \vee a) = (p \wedge a) \vee (p \wedge x)$ for some $p \in P$ and $a \in \mathcal{L}_1$ which gives $p \wedge x = 0 \in K_1$, as $K_1$ is a filter, a contradiction. Thus $\mathbf{m}$ is a maximal filter of $\mathcal{L}$. Similarly, $\mathbf{m}'$ is a maximal filter of $\mathcal{L}$. The in particular statement is clear. □

Next, we justify that decomposable lattices have no $J$-filters.

**Theorem 4.17.** *Let $\mathcal{L} = \mathcal{L}_1 \times \mathcal{L}_2$ be a decomposable lattice. Then $\mathcal{L}$ has no $J$-filters.*

*Proof.* Let $F = F_1 \times F_2$ be a $J$-filter of $\mathcal{L}$, where $F_1$ is a filter of $\mathcal{L}_1$ and $F_2$ is a filter of $\mathcal{L}_2$. Since $(1,1) = (0,1) \vee_c (1,0) \in F$ and $(0,1), (1,0) \notin J(\mathcal{L})$ by Lemma 4.16, we conclude that $(0,1), (1,0) \in F$ and so $F = \mathcal{L}$ which is impossible. Thus $\mathcal{L}$ has no $J$-filters.

□

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